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THESIS

THE OPTIMUM ALLOCATION OF AIRSTRIKES
AGAINST A TRANSPORTATION NETWORK
FOR AN
EXPONENTIAL DAMAGE FUNCTION

by

Richard Oliver Nugent

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ABSTRACT

A procedure is presented for solving the problem of allocating airstrikes for the interdiction of a transportation network when the flow of supplies is limited by the capacity of the system. The damage function is assumed to be exponential.

The inputs required are the upper and lower bounds on the capacity of each arc, the vulnerability of the arc to attack and the number of aircraft sorties available for the mission. The procedure determines the segments in the network to attack as well as the level of attack to reduce the network flow capacity to a minimum.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

Figure

TABLE OF SYMBOLS AND ABBREVIATIONS

(i,j)	an arc between nodes i and j
U_{ij}	maximum length (capacity in the primal) of arc (i,j)
L_{ij}	a lower bound on the length (capacity) of arc (i,j)
W_{ij}	the vulnerable length (capacity)
k_{ij}	the number of sorties allocated to arc (i,j), $k_{ij} \geq 0$
K	the resource constraint, or total units of effort available
b_{ij}	a measure of the vulnerability of arc (i,j), or a measure of the efficiency of the weapon system used to attack arc (i,j)
s_{ij}	the surviving portion of the vulnerable length
m_{ij}	the resulting length of arc (i,j) when k_{ij} sorties are allocated.
n	the total number of arcs in a route from source to sink
N	the total number of nodes in the network; the sink
x_{ij}	actual flow in arc (i,j) in the primal
u	the Lagrange multiplier
$h(u)$	$\sum_{W_{ij}b_{ij}>u} (1/b_{ij}) \ln(W_{ij}b_{ij}/u)$
u^*	the value of u such that $h(u) = k$
v_q	the q^{th} largest value of $W_{ij}b_{ij}$ on a route
M^*	the minimum feasible length of the route being considered
c_{ij}	proportional damage parameter
r	the number of distinct routes from source to sink without loops

I. INTRODUCTION

A. SUPPLY NETWORK

In order to conduct sustained military operations any distance from its permanent base a military force must have some means of resupply. Quite often a major portion of this resupply comes over the land transportation system. This is particularly the case when the opposing forces may have air and/or sea supremacy or the means of resupply by sea or air are not available.

The ability of the military force to operate at various levels of combat performance may be seriously affected by the capability of the transportation system to carry the necessary supplies. Limitations on the cargo-carrying capacity of the transportation network may occur as the result of the characteristics of the road network and vehicles being used. This capacity, expressable in terms of tons per day, measures the ability of communications lines to transport cargo.

The type and amount of supplies required to sustain a military force vary greatly depending on the type of warfare being conducted as well as the level of combat. Small guerrilla forces do not require as much as larger conventional forces and may be able to provide much of their own support from the area they are working in. In general however there are military supplies which must be brought in. Logistics planning must consider the supply capacities of roads and transportations requirements to support a military operation.

The Army Field Manual for Staff Officers on Organization, Technical and Logistical Data [1] lists and describes planning factors for the logistical support of various size forces in different types of environment.

The requirements to support operations in various situations are expressed in terms of pounds-per-man-per-day and include all classes of supply.

Holliday [2] has provided the research allowing the concept of network capacity to be applied to highways. In work done for the Advanced Research Projects Agency, he provides the methodology for determining road capacities. He analyzes traffic cargo flow based on operational factors and relates the road supply capacity to the number of vehicles required to achieve that capacity.

B. INTERDICTION

When the resupply capability of one force is limited by the capacity of the transportation network over which it must operate, it is to the advantage of the opposing force to consider the interdiction of that network. It may be worthwhile diverting resources toward network interdiction in order to further reduce the capacity of the transportation system. By restricting the capacity, the flow of goods is ultimately restricted. This is taken as the objective rather than minimizing flow directly. Moving vehicles become difficult to interdict when provided with sufficient defense, jungle canopy, camouflage or cover of darkness. In addition, destroying vehicles with cargo may not accomplish the desired objective unless either vehicles or supplies are in short supply.

Airstrikes may be used to further restrict the capacities of segments in the network. This may be done by destroying bridges, ferries or railroad track, cratering or blocking roads, et cetera. In addition, aircraft may reduce the capacity flow of traffic over a network by reducing the period during which vehicles may operate.

Several authors have considered the supply network interdiction problem. Wollmer [3,4] determines the most vital links of a network and examines the effect on capacity of removing the most vital link from the network. Arcs are subject to breakdown which results in a decrease in capacity (1) by a known quantity and then (2) by a quantity which is a random variable. He finds the greatest reduction of flow possible if n breakdowns occur and where these must occur. No consideration is made for the cost of the breakdown which, in the case of interdiction, corresponds to the number of aircraft allocated. In addition his method appears cumbersome when there may be more than a few breakdowns per arc.

Durbin [5] evaluates the ability of a transportation network to deliver supplies as the road segments are successively destroyed and repaired. The methodology is provided for successively removing the next most vital arc until the flow is reduced to zero or the pre-determined number of links have been destroyed. The process then steps to the next day, restores repaired links and continues. He too uses lower bounds of zero and does not consider the cost to destroy the link.

The effectiveness of airstrikes against a segment of the network may be a function of many factors. The type of weapon system involved, its accuracy and destructiveness, characteristics of the segment such as overhead cover and concealment, defense, vulnerability and recoverability are all examples of such factors.

Mustin [6] has looked at the interdiction problem when the decrease in capacity of an arc has a fixed deterministic rate and is known; the reduction in capacity is assumed linear between upper and lower bounds with the latter being positive. He proposes a linear approximation when the returns are not constant.

Generally airstrikes can be expected to reduce the capacity of a segment of the network according to some sort of law of decreasing marginal returns. This is reasonable when it is considered that although a bridge may be destroyed, it may still be possible to ford or ferry supplies across a river. Similarly shuttling may be conducted to overcome the effect of cratered or blocked roads.

This paper therefore considers a damage function which exhibits decreasing marginal returns throughout; namely, an exponential function.

C. ALLOCATION

The general interdiction problem facing the strike planner on a given day is which segments of the network to attack and at what level. His objective is to create the maximum reduction in flow capacity in order to have the greatest effect on the opposing side's combat performance. The restriction placed upon him is the number of sorties available.

To accomplish his objective the strike planner must make the most efficient utilization of the resources available to him. He must consider not only the capacity of each segment of the network but also its vulnerability and importance. An arc with the largest capacity is not necessarily the most worthwhile target, nor is an arc with the highest vulnerability.

It is assumed that the strike planner has available to him the necessary information concerning the factors involved. He knows the upper and lower bounds on capacity of each segment and its vulnerability. The number of sorties available to him is fixed. His problem is how to allocate those sorties in order to have the greatest effect on the enemy's resupply capability.

II. OBJECTIVE AND SCOPE

The purpose of this paper is to present a method of solving the problem of allocation of effort in the interdiction of a transportation network for a damage function having decreasing marginal returns.

The strike planner is assumed to have available to him information regarding the upper and lower bounds of segment capacities. It is also assumed that he knows the vulnerabilities of each segment. Finally, the total number of sorties available is assumed known and fixed. A necessary assumption is that the various segments are independent in so far as the reduction of capacity of one arc does not affect the vulnerability of another arc. The proportional reduction in capacity of arc (i,j) with k_{ij} sorties is assumed to be $1 - \exp(-b_{ij}k_{ij})$.

The solution procedure selects the segments to be attacked which will give the smallest resulting value of maximum capacity and the number of sorties to allocate to each segment. Also given in the results is the minimum value of capacity obtainable with the number of sorties available.

III. THE MODEL

A. NETWORK DESCRIPTION

The transportation system can be represented by a network of arcs and nodes. Nodes represent intersections of road segments. Further, they may be used to represent any point at which it is convenient to distinguish between the road characteristics on either side of the node. Arcs represent road segments. Each arc joins two nodes. Associated with each arc are quantities representing the upper and lower bounds on capacity and the vulnerability of the arc. Both upper and lower bounds are in terms of the maximum amount of flow which may pass over the arc in a unit of time, say in tons per day.

Two special nodes are the source and sink. It is assumed that all goods flow from the source to the sink. If there are several sources or sinks, this may be taken care of by adding artificial nodes and arcs. The capacities of these artificial arcs are infinite and they are not vulnerable to attack.

The notation (i,j) represents the arc between nodes i and j. Nodes are numbered from 1 to N with 1 the source and N the sink. The intermediate nodes have any of the values between 1 and N. For a particular route, n will represent the total number of arcs in the route.

The actual flow in an arc (i,j) is designated as x_{ij} if it is flowing from i to j and as x_{ji} if it is flowing from j to i. Flow is assumed to be from the source to the sink but may be in either direction over the intermediate arcs and nodes. Due to the undirected structure of the network considered, capacities on arcs represent bounds on net flow over the arcs in either direction.

The principle of flow conservation at the nodes is assumed. This means that flow out of a node equals flow into a node; that is, there is no storage at nodes.

The capacity on arc (i,j) at any time is represented by m_{ij} and implies

$$0 \leq x_{ij} \leq m_{ij}.$$

As long as the capacity of an arc is the same in both directions, $m_{ij} = m_{ji}$. This will be the nature of network arcs considered.

The lettering U_{ij} and L_{ij} represent the upper and lower bounds respectively on the capacity of the arc (i,j). If there is no lower bound then $L_{ij} = 0$. The following relationship holds:

$$0 \leq L_{ij} \leq m_{ij} \leq U_{ij} .$$

The vulnerability of arc (i,j) is represented by the parameter b_{ij}. This vulnerability parameter indicates the efficiency of the particular weapon system against the arc.

The amount of capacity of arc (i,j) which is vulnerable to attack is $W_{ij} = U_{ij} - L_{ij}$. As the expected proportion of reduction is $1 - \exp(-b_{ij}k_{ij})$, the reduction function which results is then

$$W_{ij}[1 - \exp(-b_{ij}k_{ij})] .$$

W_{ij} would be the expected reduction in capacity for a very high effort; that is, for $k_{ij} = \infty$. Only by allocating an infinite (or very high amount of) effort toward arc (i,j) would the expected resulting capacity be its lower bound.

For finite k_{ij} the remaining capacity of arc (i,j) is given by:

$$m_{ij} = U_{ij} - W_{ij}[1 - \exp(-b_{ij}k_{ij}] .$$

B. MAXIMAL FLOW DETERMINATION

It is assumed that the opposing force wishes to maximize the total flow, Q, through the transportation network available. This is the standard maximal flow problem which can be stated as the following linear programming problem:

$$\text{maximize} \quad Q = \sum_k x_{1k} = \sum_j x_{jN}$$

$$\text{subject to} \quad \sum_k x_{ik} - \sum_j x_{ji} = 0 \qquad i = 2,3,\ldots, N-1$$

$$0 \leq x_{ij} \leq m_{ij} \qquad i = 1,2,\ldots,N$$

$$j = 1,2,\ldots,N \quad i \neq j$$

where $m_{ij} = 0$ if the particular arc (i,j) does not exist; otherwise m_{ij} is a known constant. The notation x_{ik} represents the flow from node i to any node k. The notation x_{ji} represents the flow from any node j into node i.

The maximal flow can be determined by use of the maximal-flow minimal-cut theorem [7]. A cut set, defined as a set of arcs which separates the source from the sink, has a value defined as the sum of the capacities of all arcs in the cut set. The maximal-flow minimal-cut theorem states that the maximal flow in a network is equal to the minimal value of all cut sets.

It is convenient to use the notion of a topological dual in finding the cut set through a network [8]. The topological dual will be a network where arcs have lengths instead of capacities. There is a one-to-one correspondence between the cuts of the original network and the routes through the dual. The shortest route through the dual corresponds

to the minimal cut set in the original or primal network; its length is equal to the value of the minimal-cut set. The topological dual is defined only for planar networks, that is, for a network where no two arcs intersect except at a node when the network is represented on a plane. This will be the only type of network considered in this paper.

To form the dual, an additional arc is drawn to connect the source and sink of the primal. This forms the modified primal, G. The dual of the primal is constructed by taking a vertex inside each region of G and one vertex outside G and connecting vertices in adjacent regions with arcs which cross arcs of the primal. The vertices are points which become the nodes of the dual. Only one arc can cross an arc of the primal and no arc crosses the artificial arc. Select as the source or sink the node outside G and then designate the node in the region created by the artificial arc of G as the other end of the network.

The lengths of the arcs in the dual are the capacities of the arcs which they intersect in the primal. Because the primal network is undirected, the arcs of the dual are also undirected. The maximal flow through the primal can therefore be determined by finding the shortest route through the dual. The shortest route is found by finding the path from source to sink such that the summation of arc lengths in the path is minimum among all possible paths from source to sink.

C. THE INTERDICTION PROBLEM

The interdiction problem associated with the dual involves determining the feasible distribution of effort over a route which will result in reducing its total length by the largest amount. The improvability of the arc in the dual is the same as the vulnerability of the primal arc. Each unit of effort (each sortie) allocated toward an arc in the dual

shortens the length of that arc resulting in the reduction of the associated arc capacity in the primal arc. Some route through the dual will have the shortest length after optimal allocation. The problem is to find this "shortest" route and determine the optimal allocation to attain its final length. Mathematically, the problem is to find that route which

$$\text{minimizes} \quad \sum_{i,j} \{U_{ij} - W_{ij} [1 - \exp(-b_{ij}k_{ij})]\}$$

$$\text{subject to} \quad \sum_{i,j} k_{ij} \leq K , \quad k_{ij} \geq 0 .$$

In the equations above and throughout this paper, the notation $\sum_{i,j}$ is taken to mean the summation over all arcs in the particular route.

The restraint of the problem above can be considered an equality as the objective function is continuous and decreasing for finite K; that is, the value of the objective function for any number of sorties less than K will always be greater than the value of the objective function for finite K.

IV. SOLUTION PROCEDURE

A. PREVIEW

By utilizing the topological dual, the problem of allocation becomes one of determining the shortest feasible route through the dual In finding the solution, there are three separate problems which must be dealt with:

1) which route through the dual to select;
2) which arcs in the route to attack;
3) how much effort to allocate to each of the attacked arcs.

The amount of computations is reduced by establishing upper and lower bounds on the feasible lengths for each route, thereby eliminating some routes from further consideration.

Although an integer solution is not provided in the calculations of the procedure, the value of the objective function obtained in the example problem was not changed greatly by using a simple round-off procedure. This is generally true due to the nature of the exponential distribution. In the event that the parameters are such as to be working on the steep slope portion of the exponential curve, it may be necessary to investigate the integer solutions.

B. ALLOCATION OF STRIKE EFFORT TO A GIVEN ROUTE

In approaching the problem, it is useful to consider a problem proposed by Koopman [9] and explained in more detail by Danskin [10].

Consider first the following lemma by Gibbs as given by Danskin:

Suppose the set $x^0 = (x_1, x_2^0, \ldots, x_n^0)$ maximizes $\sum_i f_i(x_i)$ subject to the conditions $\sum_i x_i = X$, $x_i \geq 0$. Suppose further the f_i are differentiable. Then there exists a scalar u, such that

$$f_i'(x_i^o) = u \quad \text{if } x_i^o > 0 ;$$
$$\leq u \quad \text{if } x_i^o = 0 ;$$

where f_i' is the derivative of f_i with respect ot x_i.

For the allocation of effort toward interdiction the f_i of Gibbs' lemma is $W_{ij}[1 - \exp(-b_{ij}k_{ij})]$. From the conditions of the lemma, the following relations must be satisfied for an optimal allocation on a route:

$$W_{ij}b_{ij} \exp(-b_{ij}k_{ij}) = u \quad \text{if } k_{ij} > 0 ; \tag{1}$$

$$W_{ij}b_{ij} \leq u \quad \text{if } k_{ij} = 0 . \tag{2}$$

It will be desirable to show that (1) and (2) are necessary and sufficient conditions. Suppose $W_{ij}b_{ij} \leq u$ and $k_{ij} > 0$, then (1) must hold. Because $\exp(-b_{ij}k_{ij})$ is always less than 1 for b_{ij} and k_{ij} both positive, then $W_{ij}b_{ij} > u$, which is a contradiction of the original assumptions. Therefore $k_{ij} \leq 0$. But $k_{ij} \geq 0$ is required for feasibility. Thus $W_{ij}b_{ij} \leq u$ implies $k_{ij} = 0$. This together with (2) implies

$$W_{ij}b_{ij} \leq u \quad \text{if and only if } k_{ij} = 0 . \tag{3}$$

The feasibility requirement together with (3) indicates that $W_{ij}b_{ij} > u$ implies $k_{ij} > 0$ as $k_{ij} = 0$ can only happen if $W_{ij}b_{ij} \leq u$.

The value of k_{ij}, where $k_{ij} > 0$, may be derived from (1):

$$k_{ij} = (1/b_{ij}) \ln(W_{ij}b_{ij}/u) \tag{4}$$

where u must satisfy

$$\sum_{W_{ij}b_{ij} \quad u} (1/b_{ij}) \ln(W_{ij}b_{ij}/u) = K$$

The summation sign in the above relation is taken as the summation over all arcs in the route with $W_{ij}b_{ij}$ greater than u.

To determine the value of u, first define h(u) and v_q as follows:

$$h(u) = \sum_{W_{ij}b_{ij} > u} (1/b_{ij}) \ln(W_{ij}b_{ij}/u)$$

$$= \sum_{W_{ij}b_{ij} > u} (1/b_{ij}) \ln(W_{ij}b_{ij}) - \ln(u) \sum_{W_{ij}b_{ij} > u} (1/b_{ij}).$$

$v_q = W_q b_q$ = the q-th largest value of $W_{ij}b_{ij}$ in the route.

The desired value u* is that value of u which satisfies h(u) = K.

It is helpful to look at the nature of the function h(u). For u small (near zero), h(u) is large and for u sufficiently small, h(u) is larger than any finite K. For u greater than or equal to v_1 (the maximum value of $W_{ij}b_{ij}$ on that route), h(u) is equal to zero. For u between 0 and v_1, h(u) is strictly decreasing and continuous.

To find u, the following recursive formula is developed for $h(v_q)$, where $v_n \leq v_q \leq v_1$:

$$h(v_q) = h(v_{q-1}) + [\ln(v_{q-1}) - \ln(v_q)] \sum_{i=1}^{q-1} (1/b_i)$$

where $h(v_{q-1}) = \sum_{i=1}^{q-2} (1/b_i) \ln(v_i) - \ln(v_{q-1}) \sum_{i=1}^{q-2} (1/b_i)$.

Specifically,

$$h(v_n) = \sum_{W_{ij}b_{ij} > u^*} (1/b_{ij}) \ln(W_{ij}b_{ij}/u^*)$$

$$= \sum_{i=1}^{n} (1/b_i) \ln(W_i b_i/u^*). \qquad (5)$$

The single subscript on the b's indicates the b_q corresponding to the v_q which is the q-th largest value of $W_{ij}b_{ij}$.

If $K > h(v_n)$, then there exists a $u^* < v_n$ such that

$$K = h(v_n) + [\ln(v_n/u^*)] \sum_{i=1}^{n} (1/b_i).$$

Then

$$u^* = v_n \exp\{-[K-h(v_n)]/\sum_{i=1}^{n} (1/b_i)\} .$$

In the event $K < h(v_n)$, $h(v_q)$ is successively computed for $q = n$, $n-1,\ldots,1$, that is, from the smallest to the largest value of v_q, until

$$h(v_q) \geq K > h(v_{q-1})$$

for some q. Then the desired value of u is

$$u^* = v_{q-1} \exp \{-[K-h(v_{q-1})]/\sum_{i=1}^{q-1} (1/b_i)\} .$$

If $h(v_q) = K$, then $u^* = v_q$.

The value of k_{ij}, the amount of effort to allocate for an arc having $W_{ij}b_{ij} > u^*$, may be determined from (4):

$$k_{ij}^* = (1/b_{ij}) \ln(W_{ij}b_{ij}/u^*). \tag{6}$$

When $W_{ij}b_{ij} \leq u^*$, $k_{ij}^* = 0$, as determined in the derivation.

It remains to determine which route in a network to select in order to get the minimum feasible route through the network. Letting

$$M^* = \sum_{i,j} \{U_{ij} - W_{ij}[1 - \exp(-b_{ij}k_{ij})]\}$$

for a particular route after the k_{ij}^* have been determined, the problem is to find that route for which M^* is minimum over all possible routes.

For this a stepwise procedure is developed to eliminate some of the routes from consideration by means of establishing bounds on the feasible route lengths.

For the special case of $L_{ij} = 0$, a simplification results because $W_{ij} = U_{ij}$. The criterion for attack of arc (i,j) is that $U_{ij}b_{ij}$ is greater than u^*. This can be handled by direct substitution and presents no problem in the computations.

C. STEPWISE PROCEDURE

1) Determine the minimum normal route using the upper bounds on lengths, U_{ij}. This may be done using any of the shortest route algorithms [11]. The route through the network is found such that the sum of the upper bounds is the minimum over all routes. This then becomes the first least upper bound for the network.

2) Determine the r-shortest routes through the network using the lower bounds on capacities, where r is the number of distinct routes in the network from source to sink without loops.

Because of the nature of the problem it is not necessary to consider any route with a lower bound greater than the least upper bound. In general, therefore, the number of routes that have to be considered will be considerably less than the total number of distinct routes in the network. In addition, there is no practical reason for passing over an arc or node more than once because, in determining a minimum cut set, if M^* is the length of a feasible route without loops, any route considering loops would have a length greater than M^* and need not be considered. Therefore if some computational procedure is used to determine the r-th shortest route, caution must be taken that no arc or node is included twice in any route. Clarke, Krikorian and Rausen [12] provide a procedure for

determining the N best loopless paths in a network where N is any positive integer.

3) Beginning with the route having the smallest bound, determine improved bounds by computing $h(v_n)$ from equation (5).

4) As $h(v_q)$ is a continuous and decreasing function, if K is greater than $h(v_n)$, this means that u^* is less than v_n and all k_{ij} are positive. The surviving portion of the vulnerable capacity is

$$W_{ij} \exp(-b_{ij}k_{ij}) = u^*/b_{ij} < v_n/b_{ij} ,$$

and the feasible minimum length for the route is

$$M^* = \sum_{i,j} L_{ij} + u^* \sum_{i,j} 1/b_{ij} .$$

The value of u^* can now be computed in closed form and the value of the feasible minimum length for the route obtained. The equation for computing u^* is

$$u^* = v_n \exp \{-[K - h(v_n)]/ \sum_{i,j} (1/b_{ij})\} .$$

5) If K is less than or equal to $h(v_n)$, then u^* is greater than or equal to v_n. A lower bound on the feasible route length is thus provided:

$$L.B. = \sum_{i,j} L_{ij} + v_n \sum_{i,j} (1/b_{ij}) \le \sum_{i,j} L_{i,j} + u^* \sum_{i,j} (1/b_{ij}) = M^*$$

This bound can prove useful in the following manner: if any route has a minimum feasible length or lower bound which is greater than the least upper bound for the network, this route can be eliminated from further consideration.

6) If any feasible route length is less than the least upper bound for the network, this becomes the new network least upper bound.

7) Only those routes with lower bounds less than or equal to the least upper bound of the network need be considered further.

8) To continue, it is now necessary to perform the summations over the arcs which have $W_{ij}b_{ij}$ greater than u^* for those routes for which u^* and the minimum feasible length have not yet been determined. Beginning with the route having the next larger lower bound, order the v_q in increasing order, where

$$v_q = W_q b_q = \text{the q-th largest value of } W_{ij}b_{ij} \text{ in that route;}$$

$$v_1 = W_1 b_1 = \max_{i,j} W_{ij}b_{ij} ;$$

$$v_n = W_n b_n = \min_{i,j} W_{ij}b_{ij} .$$

Summations are henceforth made over the single subscript. Successively compute $h(v_q)$ for $q = n,n-1,\ldots,1$ for the route being considered until

$$h(v_q) \geq K > h(v_{q-1}) \quad \text{for some } q \quad 1 \leq q \leq n .$$

9) At each iterative computation, if it is found that $h(v_q) > K$, a new improved lower bound is provided as this means another arc has been found for which no sorties will be allocated and that $u^* > v_q$. This means the feasible route length will not be as great as if $u = v_q$, that is,

$$L.B. = \sum_{i=q}^{n} U_i + W_q b_q \sum_{i=1}^{q-1} (1/b_i) + \sum_{i=1}^{q-1} L_i \leq M^* . \qquad (7)$$

If the lower bound for a route exceeds the current least upper bound for the network at any time, this route may be dropped from further consideration.

10) Once the value of q is found so that both $h(v_q) \geq K$ and $K > h(v_{q-1})$, the improved lower bound is found as in (7), and the least upper bound for the route is

$$U.B. = \sum_{i=q}^{n} U_i + \sum_{i=1}^{q-1} L_i + W_{q-1} b_{q-1} \sum_{i=1}^{q-1} (1/b_i) \geq M^* . \quad (8)$$

The above relation is a result of the fact that K greater than $h(v_{q-1})$ implies that u^* is less than v_{q-1}.

11) The minimum feasible route length can now be derived for the route being considered:

$$M^* = \sum_{i=q}^{n} U_i + \sum_{i=1}^{q-1} L_i + u^* \sum_{i=1}^{q-1} (1/b_i)$$

where

$$u^* = v_{q-1} \exp\{ -[K - h(v_{q-1})] / \sum_{i=1}^{q-1} (1/b_i) \}.$$

12) The process is repeated from step 6 until all routes have been considered. The route with the minimum feasible route length over all routes in the network is the one to allocate airstrikes to. Ties indicate that there are two or more minimum feasible cut sets in the primal.

13) Having selected the route which produces the minimum feasible distance through the network and the u^* for that route, the remainder of the solution is available. Those arcs with $W_{ij} b_{ij}$ greater than u^* will be attacked and the level of attack on arc (i,j) will be k_{ij}^* from equation (6).

D. EXAMPLE

The following is a hypothetical example of a transportation network. The first diagram in figure 1 is the primal network. Flow is from node 1 to node 6.

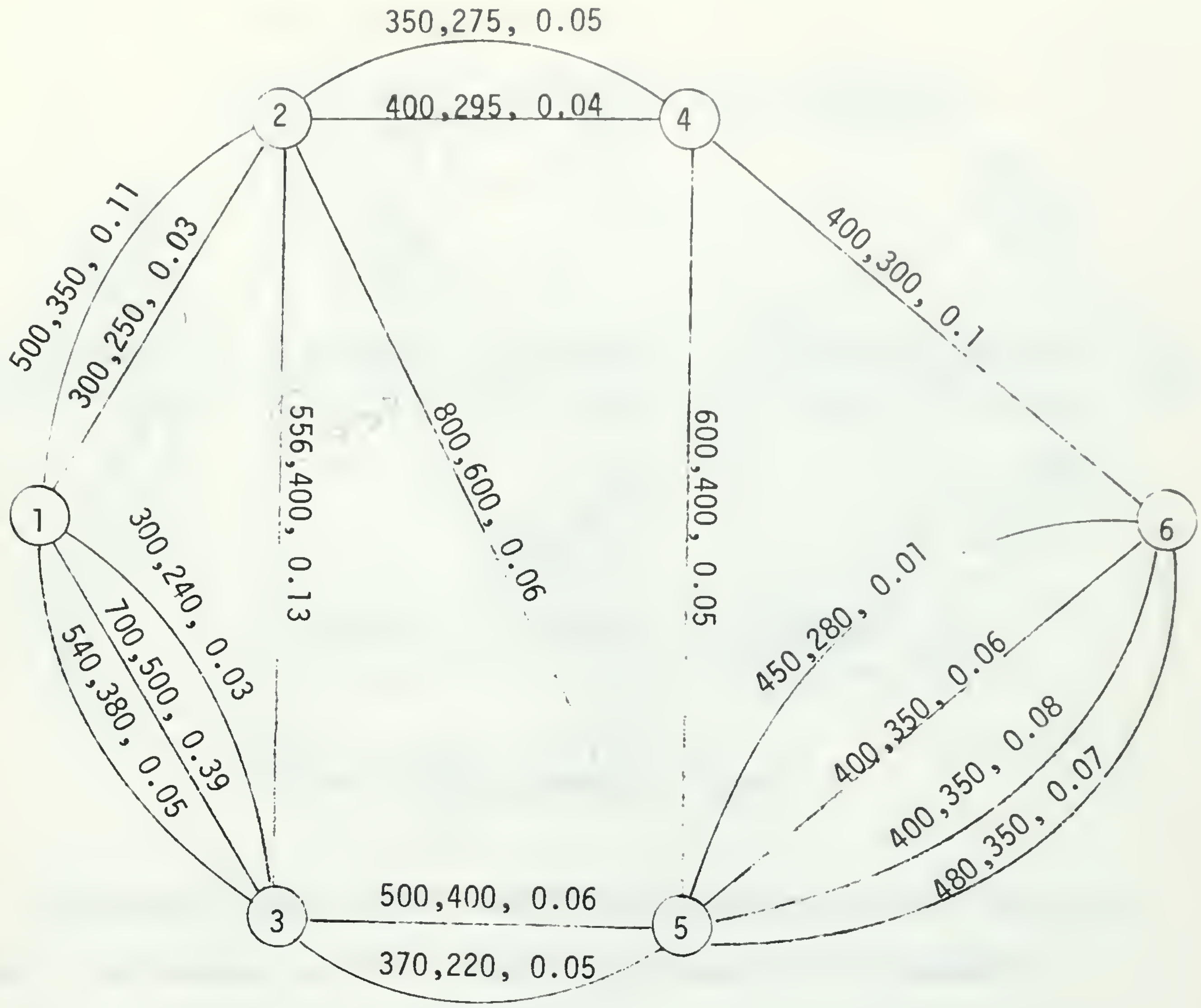

Figure 1. An Example Network

There are three numbers associated with each arc: U_{ij}, L_{ij} and b_{ij}. These represent the upper and lower bounds and the vulnerability parameter, respectively, of each arc. The number of sorties available for the interdiction of the network is 40.

In figure 2 the topological dual is formed by first constructing the artificial arc between the source and the sink. Asterisks are used to denote the nodes of the dual.

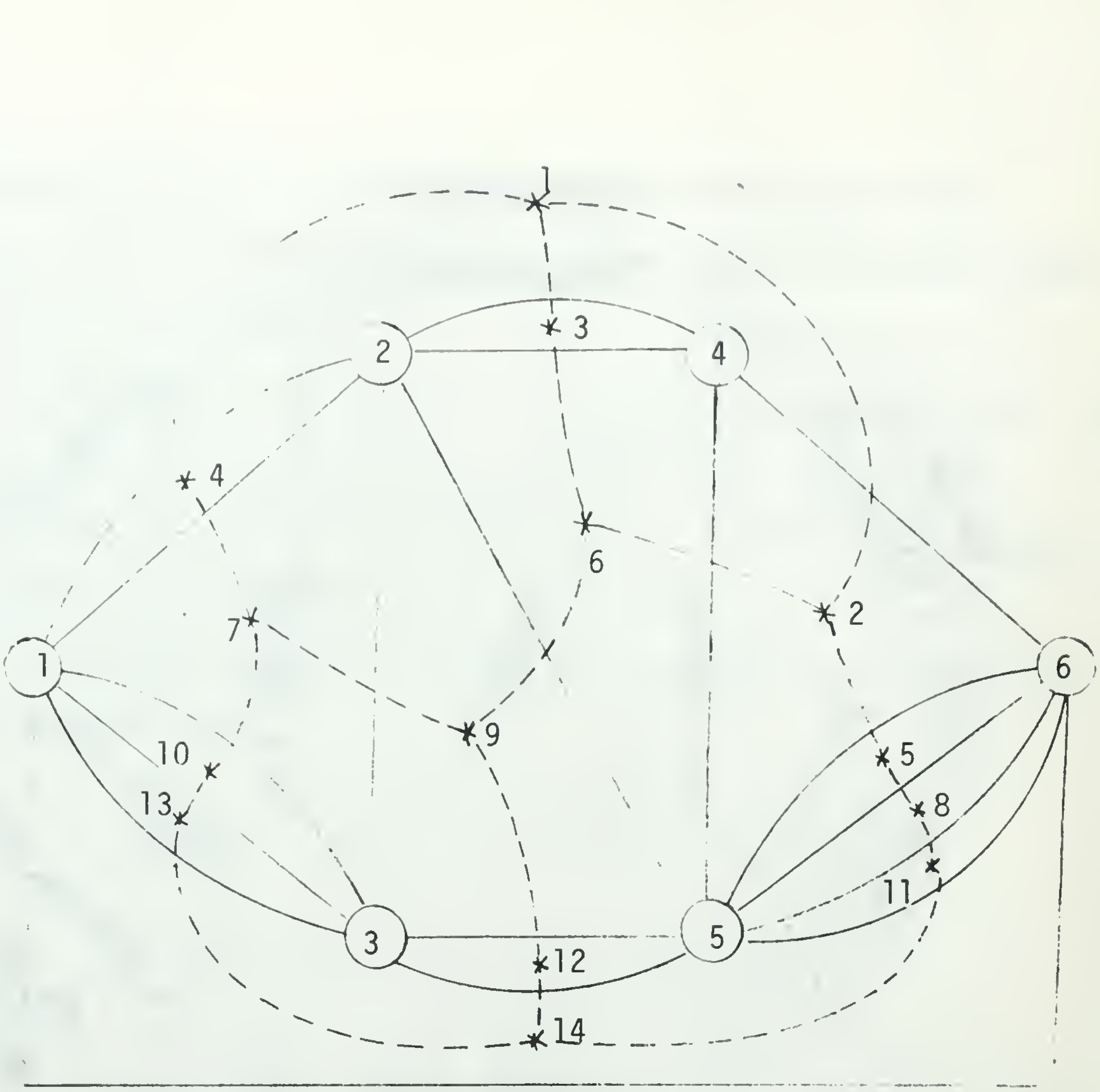

Figure 2. Construction of the Dual

Figure 3 shows the topological dual with the appropriate figures transferred from the primal. For convenience of reference the various routes between the source and sink are designated by the nodes over which they pass.

R_1: 1,2,5,8,11,14	R_6: 1,2,6,9,7,10,13,14
R_2: 1,2,6,9,13,14	R_7: 1,3,6,2,5,8,11,14
R_3: 1,3,6,9,13,14	R_8: 1,3,6,9,7,10,13,14
R_4: 1,4,7,9,13,14	R_9: 1,4,7,9,6,2,5,8,11,14
R_5: 1,4,7,10,12,14	

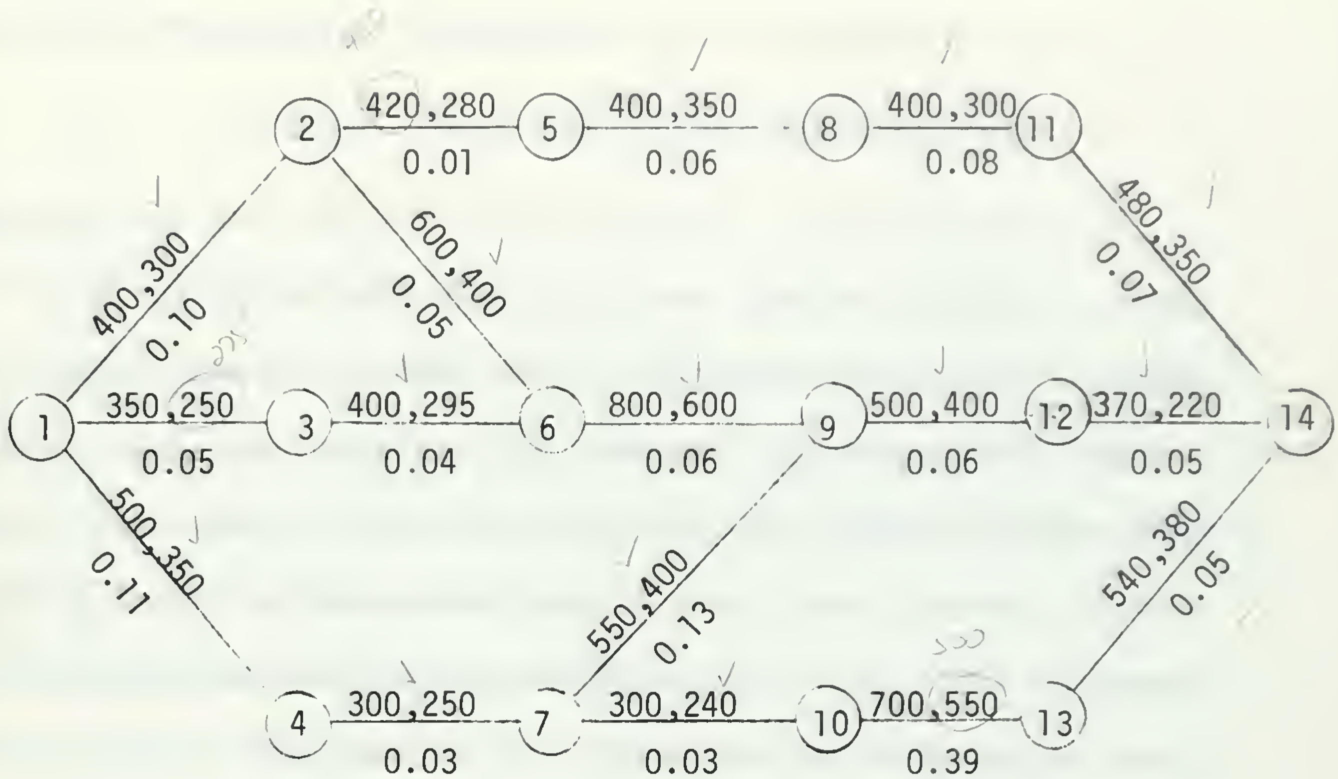

Figure 3. The Topological Dual

after correction

The minimum length normal route is provided by R_1 with a length of 2130. This becomes the first least upper bound for the network.

In determining the r shortest routes using the lower bounds, the following order is found:

R_1: 1580, R_4: 1620, R_5: 1770, R_3: 1790, R_2: 1920, R_7: 2250

The lower bound on the next shortest route is from R_7; it is 2250 which exceeds the present least upper bound. For this reason it is not necessary to consider R_6, R_7, R_8 or R_9.

Computations are begun on the routes in increasing order of lower bounds. When a feasible minimum length is found which is lower than the third lower bound of the remaining routes, the optimum solution has been found. No other route can be reduced to this amount.

The computations for R_1 are done first. The ordered v_q for R_1 are:

$v_1 = 0.1(100) = 10$; $v_2 = 0.07(130) = 9.1$; $v_3 = 0.08(100) = 8$;

$v_4 = 0.06(50) = 3$; $v_5 = 0.01(170) = 1.7$.

The function $h(v_5)$ is computed, and it found that the number of sorties available is less than $h(v_5) = 70.51$. Similarly it is found that $K = 40$ is less than $h(v_4) = 40.15$, but K is greater than $h(v_3) = 4.07$. This indicates that arcs (1,2), (11,14) and (8,11) would be attacked while arcs (2,5) and (5,8) would have no sorties allocated. A first improved lower bound is obtained by adding the sum of the upper bounds of those arcs which would not be attacked and the sum of the lower bounds of those arcs which would be attacked if this route were chosen.

$$L.B. = \sum_{i=4}^{5} U_i + \sum_{i=1}^{3} L_i = 1800$$

A second improved lower bound is obtained using equation (7)

$$L.B. = \sum_{i=4}^{5} U_i + \sum_{i=1}^{3} L_i + v_4 \sum_{i=1}^{3} (1/b_i)$$

$$= 1800 + 3(36.79) = 1910.37$$

From (8) an upper bound for R_1 is

$$U.B. = \sum_{i=4}^{5} U_i + \sum_{i=1}^{3} L_i + v_3 \sum_{i=1}^{3} (1/b_i)$$

$$= 1800 + 8(36.79) = 2094.32$$

This becomes the new least upper bound for the network and replaces the figure 2130 previously derived.

The minimum feasible length for R_1 is

$$M^* = \sum_{i=4}^{5} U_i + \sum_{i=1}^{3} L_i + u^* \sum_{i=1}^{3} (1/b_i)$$

where u^* is the solution to $h(u) = K$,

$$u^* = v_3 \exp[-(40 - 4.0719)/36.7857] = 3.0125 .$$

Therefore

$$M^* = 850 + 950 + 3.01(36.79) = 1910.74 .$$

This is less than the current least upper bound and therefore becomes the new least upper bound for the network. R_2 may now be eliminated from further consideration as it has a current lower bound of 1920. Only R_3, R_4 and R_5 remain for consideration.

The computations for R_4 result in

$$\text{L.B.} = 1750.17 ,$$

$$\text{U.B.} = 1990.70 ,$$

$$u^* = 4.34 ,$$

$$M^* = 1902.14 .$$

R_4 is therefore better than R_1. Only R_3 and R_5 remain for consideration as possible best routes. Computations for R_5 result in finding an improved lower bound of 1936 and R_5 can be eliminated. Similarly the improved lower bound for R_3 is 2159 and R_3 need not be considered further. R_4 is therefore the best route. The non-integer solution for the number of sorties to allocate to each arc is

$$k_{1,2} = 12.13 ,$$

$$k_{2,5} = 0 ,$$

$$k_{5,8} = 11.55 ,$$

$$k_{8,11} = 5.39 ,$$

$$k_{11,14} = 10.93 .$$

The optimal integer solution is

$$k_{1,2} = 12 ,$$

$k_{2,5} = 0$,

$k_{5,8} = 12$,

$k_{8,11} = 5$,

$k_{11,14} = 11$,

which results in a feasible length of 1902.21 . This is better than the non-integer solution of the closest contender, R_1 .

Returning to the primal, the cut set to be attacked is as shown in figure 4.

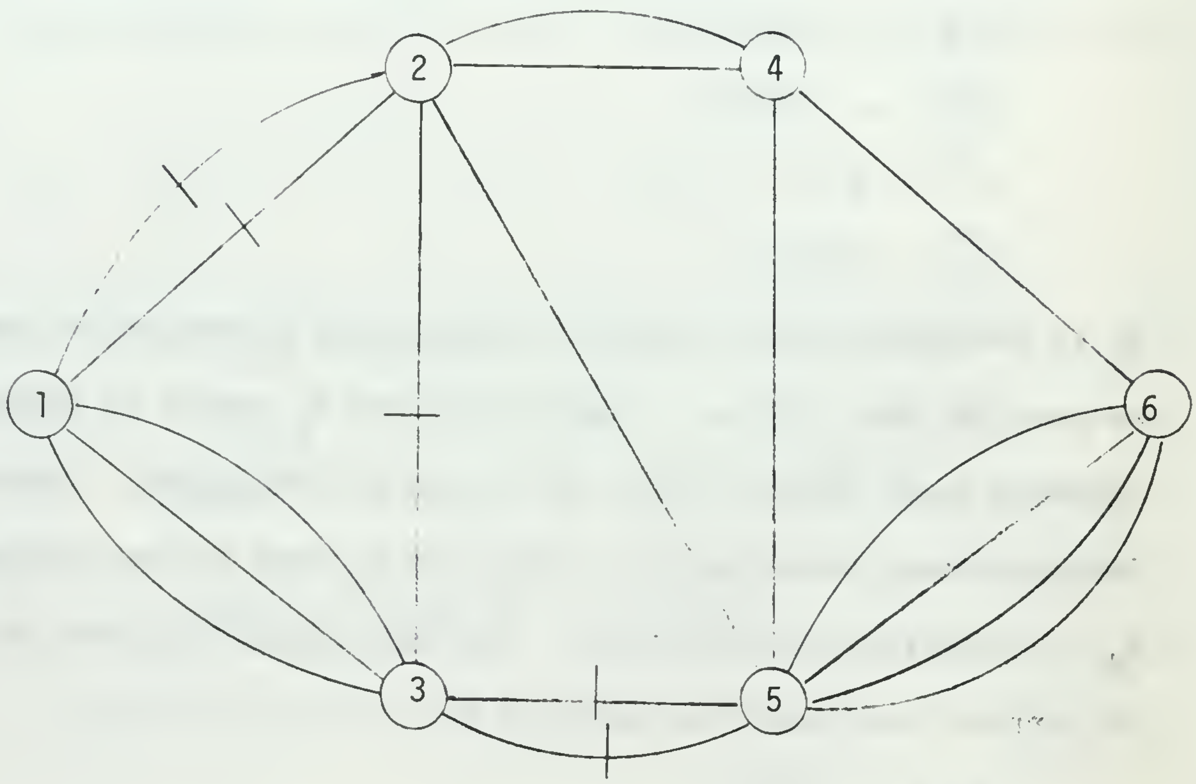

Figure 4. Minimum Cut Set After Attack

V. DISCUSSION

A. FINITENESS

The procedure described ends in a finite number of steps as the routes are systematically eliminated by means of upper and lower bounds, after which the worst possibility is complete enumeration for the remaining routes.

A non-integer solution is produced. If the parameters and the resource constraint are such as to be on the steep slope portion of the curve for the objective function, the integer solution may be vastly different than the non-integer solution. As the improved bounds in the procedure are for the feasible non-integer solution, it may be that the integer solution will actually be outside the non-integer bounds. If, however, the problem is such as to be on a less steep portion of the curve, the integer and non-integer solutions may vary only slightly. In this case a simple round-off procedure might be used to get an integer solution without much difference with the correct optimal integer solution. It appears that the only method to derive the optimal integer solution is by investigation around the non-integer solution.

B. ASSUMPTIONS

The assumptions that the values of maximum capacity, vulnerable proportions, lower bounds and vulnerability factors are known and deterministic is difficult to justify. Experimental testing such as that suggested by Holliday [13] might be used to determine whether the exponential distribution could validly be used as a damage function. It would then be necessary to determine the probability of a target hit and the

proportion of capacity reduction done by a sequence of target hits under various conditions of road types, concealment, weather and other characteristics as might be judged important.

C. INTERPRETATION OF THE LAGRANGE MULTIPLIER

The Lagrange multiplier u has an interesting interpretation and consequent use. It is the differential rate of damage for an additional sortie. In this sense, it can be interpreted as the implicit marginal price which a commander is placing on the expected destruction of one more unit of network capacity when he limits the number of sorties available for network interdiction. Another manner in which it might be used is when a commander decides he will allocate strikes up to the point where he starts getting less than u units of capacity reduction for each strike. In this sense he is making a trade-off with the other alternatives available to him. He allocates aircraft to tactical interdiction up to the point where the marginal return is in some sense the same as that of an additional aircraft allocated to any other mission.

The solution to such a problem can be derived by using the given value of u in the formulae as the criterion for attack. Those arcs with $W_{ij}b_{ij}$ greater than u would be attacked at the level k_{ij} where k_{ij} is determined from equation (6) using the given value of u as u^*.

D. ANOTHER JUSTIFICATION FOR USE OF THE EXPONENTIAL FUNCTION

Koopman [9] gives an interpretation of optimum destructiveness and makes a comparison with search theory wherein he gives some justification for the use of the exponential as a damage function. He points out the problem of optimum distribution of gunfire, bombs or other destructive missiles projected into an area. His presentation will be paraphrased in terms of optimum allocation between arcs of a route.

Two arcs with maximum lengths (capacities in the primal) of U_1 and U_2 have proportional vulnerable lengths of W_1/U_1 and W_2/U_2 respectively. Because of the type of bombing used, defenses, overhead cover or whatever, the bombing will be sufficiently accurate to hit the particular arc but not to hit the vulnerable capacity in the area except by chance. The vulnerability of all capacity is assumed to be the same on a particular arc. The result of n random hits on the target area $W_i, i=1,2$, is assumed to reduce the vulnerable capacity by a proportion $1 - c_i^n$ where c_i is less than 1. Thus after n hits on the vulnerable capacity the total capacity of the arc would be reduced to $U_i - W_i + c_i{}^n W_i$, the reduction being $W_i - c_i{}^n W_i$ and the proportionate reduction $(W_i - c_i{}^n W_i)/W_i$.

If there are N bombs available and no cost difference exists between dropping bombs on one arc than the other, then the decision as to how to allocate bombs between the two arcs is properly made on the basis of maximizing the expected damage to be inflicted.

If N_1 bombs are used on arc 1 and N_2 bombs on arc 2,

$$N_1 + N_2 = N \quad \text{and } N_1 \geq 0 \text{ , } N_2 \geq 0 \text{ .}$$

The probability that one bomb dropped on the first arc will hit the target is W_1/U_1 , assuming that the vulnerable capacity is uniformly distributed. (It is also possible to use as the probability of hit, p_i, instead of the ratio W_i/U_i.) If N_1 bombs are dropped on arc 1, the probability that R_1 of them will hit the target is (by the binomial distribution) equal to

$$\binom{N_1}{R_1}\left(\frac{W_1}{U_1}\right)^{R_1}\left(1 - \frac{W_1}{U_1}\right)^{N_1-R_1}$$

The proportional damage done by the R_1 hits being $1 - c_1^{R_1}$, the expected proportional damage produced by N_1 bombs dropped on arc 1 is given by

$$\sum_{R_1=0}^{N_1} \binom{N_1}{R_1} \left(\frac{W_1}{U_1}\right)^{R_1} \left(1 - \frac{W_1}{U_1}\right)^{N_1-R_1} \left(1 - c_1\right)^{R_1} = 1 - \exp\{-b_1N_1\}$$

where $b_1 = - \ln[1 - W_1(1 - c_1)/U_1]$.

Thus the expected proportional damage is $1 - \exp(-b_iN_i)$. The expected reduction in capacity over the route when N_1 bombs are dropped on arc 1 and N_2 bombs are dropped on arc 2 is

$$W_1[1 - \exp(-b_1N_1)] + W_2[1 - \exp(-b_2N_2)] . \quad (9)$$

The resulting expected capacity is

$$U_1 - W_1[1 - \exp(-b_1N_1) + U_2 - W_2 \exp(-b_2N_2)] \quad (10)$$

or equivalently

$$L_1 + W_1 \exp(-b_1N_1) + L_2 + W_2 \exp(-b_2N_2) . \quad (11)$$

Similar results are obtained by either maximizing (9) or minimizing either (10) or (11) subject to the restrictions that $N_1 + N_2 = N$ and $N_1 \geq 0$, $N_2 \geq 0$.

The bomb allocation problem is therefore to find the bombing strategy which gives the minimum expected value of the minimum feasible cut set through the flow network.

E. APPLICATIONS

Due to the particular characteristics of the transportation networks used in most of the land masses in Southeast Asia, the proposed procedure

might be quite appropriate for use in the interdiction problem in such areas. The fact that large portions of roadways in that area of the world are under a jungle canopy or can be easily provided with artificial concealment makes it difficult to obtain accurate bomb hits. This situation lends itself to a probabilistic treatment.

F. RECOMMENDATIONS FOR FURTHER STUDY

An analysis of actual strike data is appropriate before the exponential damage function can be considered for real-world use. Research such as that done by Holliday for ARPA [2,13] might provide such data. It would be desirable to determine hit probabilities and the expected reduction in capacity resulting from a sequence of hits by various weapon systems.

The exponential function is particularly easy to use because it provides numerous simplifications to the computational problem. Gibbs' lemma however applies to any continuous differential function with decreasing marginal returns. The data might suggest that the exponential is inappropriate but that some other function having such marginal returns can be used.

On the surface there appears to be some optimal criterion for the rounding-off of the non-integer solution. The branch-and-bound method does not seem to lend itself to a simple application in this case.

As with many other treatments involving allocation, this paper was unable to deal with the cost of lost aircraft. In a way, this was taken care of in the vulnerability parameters of the various arcs. If the probability of getting through the defenses is 3/4 and the probability of a target hit is 1/3 for those aircraft getting through the defenses, then

the probability of a target hit by a sortie allocated can be expressed in terms of conditional probability as

$$1/3 \times 3/4 = 1/4 .$$

VI. SUMMARY

A procedure has been presented for computing the optimum allocation of aircraft sorties toward the tactical interdiction of a transportation network when the flow of supplies is restricted by the capacity of the network. A specific damage function is assumed which requires knowledge or estimates of the upper and lower bounds of capacities as well as the vulnerability of each arc to the weapon system involved.

The method utilizes the notion of the topological dual and finds the minimum feasible route through the dual. The total length of this route corresponds to the minimum value to which the capacity of the network can be reduced given the number of sorties available.

The alternative routes through the dual are first screened systematically by means of bounding the minimum feasible length. The stepwise procedure eliminates those routes from consideration which cannot meet the first least upper bound. The Lagrange multiplier is used as a threshhold criterion based on the product of the parameters W_{ij} and b_{ij} to determine whether an arc should be hit. The number of sorties to allocate to the arc is then determined. The routes are considered in increasing order of sums of lower bounds. Once a feasible length or least upper bound is found less than the next lower bound, all other routes may be eliminated from consideration. The route with the minimum feasible length is the best route and corresponds to the minimum cut set in the primal.

In order to begin the stepwise procedure the inputs required are the upper and lower bounds on arc capacity, the vulnerability of each arc, and the total number of sorties available for interdiction of the network. The procedure determines which arcs to hit and at what level to achieve the minimum feasible capacity.

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13. ABSTRACT

A procedure is presented for solving the problem of allocating airstrikes for the interdiction of a transportation network when the flow of supplies is limited by the capacity of the system. The damage function is assumed to be exponential.

The inputs required are the upper and lower bounds on the capacity of each arc, the vulnerability of the arc to attack and the number of aircraft sorties available for the mission. The procedure determines the segments in the network to attack as well as the level of attack to reduce the network flow capacity to a minimum.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Interdiction						
Networks						
Allocation						